1. Does the series $\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2 n}$ converge?

This is open. The series converges if π has no unusually good rational approximations. So, probably yes.

2. It is unknown whether there are infinitely many primes of the form $2^n - 1$. But are there at least infinitely many primes which *divide* numbers of the form $2^n - 1$?

The answer is yes. It is not hard to see that if p and q are coprime, then $2^p - 1$ and $2^q - 1$ are also coprime. Therefore, each member of the set $\{2^p - 1 : \ p \text{ prime}\}$ has a distinct prime factor.

Alternately, one can show that when $p > 2$ is prime, $2^p - 1$ is not divisible by any primes less than $2p$. This is a good number theory exercise.

3. How many nonisomorphic graphs are there on 20 vertices?

This is open, by virtue of being really hard to count. Recently the answer for 19 was calculated: 24637809253125004524383007491432768. Maybe we can push it up to 20 with Brian's laptop.

4. Does every triangle have a periodic billiard ball trajectory? (Assume for simplicity that the billiard ball never hits the corners exactly.)

This is open. Here is a paper with some progress. Quote: "Our approach to the Triangular Billiards Conjecture is a bit like trying to ride a bicycle to the North Pole."

5. A matrix has rank n. We square all the entries. Can the resulting rank be arbitrarily large?

The answer is no. Say the original rows are generated by $v_1, \dots, v_n$; then the squared rows are generated by all $v_i \times v_j$, where $\times$ here means elementwise multiplication.

6. Can every sufficiently large positive integer be written as a sum of six nonnegative cubes?

This is open. It is a special case of Waring's Problem. 7 cubes suffice knows itself. 4 cubes suffice conjectures itself.

7. What is the millionth decimal digit of the $10^{10^{10^{10}}}$th prime?

The answer is 5. This was computed on stackexchange here.

8. Let $\mathcal{P}$ be a nonconstant polynomial with integer coefficients. Must there exist an integer n such that $\mathcal{P}(n)$ has at least 7 prime factors?

The answer is yes. Indeed we can replace "7" by any integer. Proof: first, I claim that if we can find 7 outputs of $\mathcal{P}$, each divisible by a different prime, then we are done. This follows from the Chinese Remainder Theorem and the fact that $\mathcal{P}(n) \pmod{p}$ is periodic with period p. Second, for any six primes $p_1, \ldots, p_6$, the set of numbers divisible only by those six primes grows faster than any polynomial. Hence we can always find 7 outputs of $\mathcal{P}$ divisible by 7 different primes. So, the answer is yes.

9. Does there exist a rectangular prism such that all the distances between vertices are integers?

This is open. Yuck.

10. Is it possible to raise a trancendental number to a trancendental power, and get an algebraic but non-rational answer?

The answer is yes. Pick an uncomputable number x, and let y be the solution to $x^y = \sqrt{2}$. Then y is uncomputable, because otherwise we could use it to compute x. If a number is uncomputable, then it must be trancendental.

11. Are there any integer solutions to $x^3 + y^3 + z^3 = 33$?

This is open for 33, 42, and 74. I bet there's a solution using really large numbers ($>10^{10}$). Brian, can you write a LaTeX program to check?